

Magnetic translation and Berry's phase factor through adiabatically rotating a magnetic field

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Abstract

For a spin subjected to an adiabatically changing magnetic field, the "solid angle result" as embodied by a rotation operator is the only non-trivial factor in the quantum evolution operator. For a charged particle, the infinite degeneracy calls for a rigorous investigation. We find that in this case, it is the product of the rotation operator and a magnetic translation operator that enters into the quantum evolution operator. This result agrees with the fact that the instantaneous hamiltonian is invariant under magnetic translation as well as rotation. Simple experimental verification is proposed.

1 Introduction

The purpose of this paper is to study the quantum mechanical problem of a charged particle moving in an adiabatically rotating magnetic field.

In the proof of the quantum adiabatic theorem as presented in Messiah's book [1], the quantum evolution operator is constructed as the product of a path-dependent factor and a dynamical factor. With the discovery of the Berry phase phenomenon [2], it is clear that the path-dependent factor in the evolution operator has nontrivial consequences; namely, after a cyclic change of the parameters, the path dependent factor is in general not equal to the identity operator but rather should recover a Berry phase factor for an adiabatic eigenstate of the Hamiltonian. When a finite fold degeneracy is involved, there is the non-Abelian generalization of the Berry phase concept due to Wilczek and Zee [3]. It remains to be rigorously investigated as to why and how the quantum adiabatic theorem holds for an infinitely degenerate system.

In the following, we choose a harmonic oscillator potential in the (changing) direction of the magnetic field. The purpose is to get rid of the unbounded motion along that direction. The infinite degeneracy is still retained because of the infinite degeneracy of the Landau levels. For the case when there is no potential in the magnetic field direction, we point out the complexity of the problem at the end of the paper; whether there is interesting result for this case is a question that may deserve further discussion.

The vector potential for a uniform but slowly rotating magnetic field can be written as

$$\mathbf{A}(\epsilon t) = \frac{1}{2}B\mathbf{n}(\epsilon t) \times \mathbf{x}, \quad (1)$$

where B is constant and $\mathbf{n}(\epsilon t)$ is a unit vector representing the direction of the magnetic field. Following Messiah [1], the time dependence of the parameters and therefore the Hamiltonian is through a slowness parameter $\epsilon = \frac{1}{T}$, where T is the duration of the adiabatic process. An adiabatic process means that T is much larger than all the other time scales involved. We have the following Hamiltonian:

$$H = \frac{1}{2m}(\mathbf{P} - \frac{q}{c}\mathbf{A}(\epsilon t))^2 + \frac{1}{2}m\omega^2(\mathbf{n}(\epsilon t) \cdot \mathbf{x} - a)^2, \quad (2)$$

where a represents the equilibrium position of the oscillator potential along the direction of $\mathbf{n}(\epsilon t)$.

A solenoid slowly rotating about a fixed point on the symmetry axis generates a magnetic field inside the solenoid that is described by the vector potential (1). Equation (1) singles out a unique point in space, $\mathbf{x} = \mathbf{0}$, where the induced electric field $-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$ vanishes. Such a point corresponds to the fixed point about which the solenoid rotates, and $\mathbf{n}(\epsilon t)$ is along the direction of the symmetry axis. There are also higher order induced electro-magnetic field due to the slow change of $-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$ and so on that are not included by (2). However, such effects are of the order of $\frac{1}{T^2}$ and therefore do not accumulate in the adiabatic process with $t \in [0, T]$. It should also be noticed that potential (1) describes the region inside the solenoid only.

We shall obtain the quantum evolution operator corresponding to (2) in the adiabatic limit of $T \rightarrow \infty$ as the product of a path-dependent geometrical operator and a usual dynamical operator. The path considered here is the path of \mathbf{n} on the two-dimensional sphere.

Observe that a Berry phase factor is often associated with an adiabatic eigenstate of the Hamiltonian. But it is equivalent to focus on the geometrical operator because Berry's phase factor is obtained at the end of the cycle by letting the geometrical operator act on an eigenstate of the initial Hamiltonian. The non-Abelian generalization [3] of the Berry phase factor can also be recovered from matrix elements of the geometrical operator among the degenerate eigenstates. Therefore, the Berry phase factor can be defined as the geometrical operator when \mathbf{n} returns back to $\mathbf{n}(0)$. Of course, the geometrical operator is more general and applies for all $t \in [0, T]$ even when $\mathbf{n}(1)$ is not equal to $\mathbf{n}(0)$.

We will find that, in the context of model (2), a magnetic translation operator [4, 5, 6] plays a natural role in addition to the rotation operator well known from Berry's spin example [2]. It will be shown that the path-dependent factor in the quantum evolution operator is exactly the product of the rotation operator and a path-ordered magnetic translation operator.

Our method is to solve the Heisenberg equations first. We will find an operator U (in a factorised form) such that any physical observable $O(t)$ is given by $O(t) = U^\dagger(t)O(0)U(t)$. This U is the evolution operator up to a phase factor. This is because any $e^{i\alpha(t)}U$ where $\alpha(t)$ is a real number also solves the Heisenberg equations. By Schur's lemma, this is also the only ambiguity. To determine the evolution operator uniquely, we must make use of the Schrödinger picture. The quantum evolution operator is finally determined in a factorised form in equation (50).

2 Solving Heisenberg Equations Using an Operator

In this section we solve the Heisenberg equations for \mathbf{x} and \mathbf{P} in the adiabatic limit by using an operator $U(t) = R(\epsilon t)M(\epsilon t)D(t)$, such that $x_i(t) = U^\dagger(t)x_i(0)U(t)$, $P_i(t) = U^\dagger(t)P_i(0)U(t)$. This $U(t)$ assumes a factorised form, where $R(\epsilon t)$ is a rotation operator, $M(\epsilon t)$ is a magnetic translation operator, and $D(t) = e^{-iH(0)t/\hbar}$.

The Heisenberg equations $\dot{x}_i = \frac{1}{i\hbar}[x_i, H]$ and $\dot{P}_i = \frac{1}{i\hbar}[P_i, H]$, upon using the commutation relations $[x_i, P_j] = i\hbar\delta_{ij}$ and $[x_i, x_j] = [P_i, P_j] = 0$, result in the following:

$$\dot{\mathbf{x}} = \frac{1}{m}\mathbf{P} - \frac{\omega_c}{2}\mathbf{n} \times \mathbf{x}, \quad (3)$$

$$\dot{\mathbf{P}} = -m\omega^2\mathbf{n}(\mathbf{n} \cdot \mathbf{x} - a) - \frac{\omega_c}{2}\mathbf{n} \times (\mathbf{P} - \frac{m\omega_c}{2}\mathbf{n} \times \mathbf{x}), \quad (4)$$

where $\omega_c = \frac{qB}{mc}$.

We shall solve the Heisenberg equations in a rotating frame specified by the unit vector $\mathbf{e}_3 = \mathbf{n}(\epsilon t)$ and two other unit vectors $\mathbf{e}_1(\epsilon t)$ and $\mathbf{e}_2(\epsilon t)$ that are determined by certain requirements (equation (7) and initial conditions). By (i) relating components of operators in this frame and the corresponding ones in the stationary frame $\{\mathbf{e}_i(0), i = 1, 2, 3\}$ using a rotation operator R and (ii) solving for \mathbf{x} and \mathbf{P} in the rotating frame $\{\mathbf{e}_i(\epsilon t), i = 1, 2, 3\}$; U is constructed. The stationary frame is just the reference frame with respect to which the Hamiltonian (2) is written. The x_i and P_i mentioned so far are components of \mathbf{x} and \mathbf{P} with respect to this frame. In the following, repeated Latin indices are summed from 1 to 3 and repeated Greek indices are summed from 1 to 2.

Let

$$\mathbf{x}(t) = \tilde{x}_i(t)\mathbf{e}_i(\epsilon t) = x_i(t)\mathbf{e}_i(0), \quad (5)$$

$$\mathbf{P}(t) = \tilde{P}_i(t)\mathbf{e}_i(\epsilon t) = P_i(t)\mathbf{e}_i(0), \quad (6)$$

where $\mathbf{e}_i(\epsilon t), i = 1, 2, 3$, are determined by the initial values $\mathbf{e}_i(0)$ and the following equation

$$\dot{\mathbf{e}}_i = (\mathbf{n} \times \dot{\mathbf{n}}) \times \mathbf{e}_i. \quad (7)$$

We demand that $\mathbf{e}_i(0)$ are unit vectors and $\mathbf{e}_3(0) = \mathbf{n}(0) = \mathbf{e}_1(0) \times \mathbf{e}_2(0)$. The meaning of equation (5) is that \mathbf{e}_i rotates with the instantaneous angular velocity $\mathbf{n} \times \dot{\mathbf{n}}$ that is perpendicular to \mathbf{n} . From equation (7) and the initial condition, we get

$$\mathbf{e}_i(0) \cdot \dot{\mathbf{e}}_j(\epsilon t) = \epsilon_{imk}(\mathbf{n}(\epsilon t) \times \dot{\mathbf{n}}(\epsilon t)) \cdot \mathbf{e}_m(0)(\mathbf{e}_k(0) \cdot \mathbf{e}_j(\epsilon t)). \quad (8)$$

Now consider the matrix E whose matrix elements are:

$$E_{ij}(\epsilon t) = \mathbf{e}_i(0) \cdot \mathbf{e}_j(\epsilon t). \quad (9)$$

By equations (8) and (9),

$$E = P \exp \int_{\mathbf{n}(0)}^{\mathbf{n}(\epsilon t)} J_m(\mathbf{n} \times d\mathbf{n}) \cdot \mathbf{e}_m(0), \quad (10)$$

where $P \exp$ means “path-ordered exponential” and where $J_m, m = 1, 2, 3$ are 3×3 matrices whose matrix elements are given by

$$(J_m)_{ik} = -\epsilon_{mik}. \quad (11)$$

The matrix E determines \mathbf{e}_i by the relation

$$\mathbf{e}_i(\epsilon t) = \mathbf{e}_k(0)E_{ki}(\epsilon t). \quad (12)$$

From equation (7) and initial conditions, it follows that $\mathbf{e}_3 = \mathbf{n}$ for all t and that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. The magnetic field \mathbf{B} becomes a constant in this frame. Moreover,

\mathbf{e}_1 and \mathbf{e}_2 can be seen as tangent vectors to the two-dimensional sphere and satisfy:

$$\dot{\mathbf{e}}_\mu \cdot \mathbf{e}_\nu = 0; \mu, \nu = 1, 2. \quad (13)$$

Therefore \mathbf{e}_1 and \mathbf{e}_2 have the geometrical meaning of being parallel transported on the 2-sphere along the path of \mathbf{n} .

If \mathbf{x} and \mathbf{P} are found in this frame, the components of \mathbf{x} and \mathbf{P} with respect to the constant frame $\{\mathbf{e}_i(0)\}$ are known through equations (5) and (6). However this is not enough, because we want to solve for x_i and P_i using an operator. So we need to find the operator that relates the rest frame operators (x_i and P_i) and the moving frame operators (\tilde{x}_i and \tilde{P}_i) which is done as follows.

Suppose $\tilde{U}(t)$ expresses the evolution of \mathbf{x} and \mathbf{P} in the frame $\mathbf{e}_i(\epsilon t)$, such that

$$\tilde{x}_i(t) = \tilde{U}^\dagger(t) x_i(0) \tilde{U}(t), \quad (14)$$

$$\tilde{P}_i(t) = \tilde{U}^\dagger(t) P_i(0) \tilde{U}(t). \quad (15)$$

Then by equation (5),

$$x_i(t) \mathbf{e}_i(0) = \tilde{U}^\dagger(t) x_i(0) \tilde{U} \mathbf{e}_i(\epsilon t). \quad (16)$$

Using the relation (12) and the linear independence of \mathbf{e}_i , we have

$$x_i(t) = \tilde{U}^\dagger(t) E_{ij}(\epsilon t) x_j(0) \tilde{U}(t). \quad (17)$$

However, $x_i(0)$ is a vector operator (first rank tensor), which means

$$x_i(t) = \tilde{U}^\dagger(t) R^\dagger(\epsilon t) x_i(0) R(\epsilon t) \tilde{U}(t), \quad (18)$$

$$R(\epsilon t) = P \exp \int_{\mathbf{n}(0)}^{\mathbf{n}(\epsilon t)} (\mathbf{n} \times d\mathbf{n}) \cdot \mathbf{e}_m(0) \frac{-iL_m(0)}{\hbar}, \quad (19)$$

where $L_m(0) = (\mathbf{x}(0) \times \mathbf{P}(0)) \cdot \mathbf{e}_m(0)$.

In fact, the tensor relation $E_{ij}(\epsilon t) x_j(0) = R^\dagger(\epsilon t) x_i(0) R(\epsilon t)$ which was used in proving (18) can be verified by comparing the derivatives of both sides with the aid of the commutation relation $[x_i(0), P_j(0)] = i\hbar \delta_{ij}$. Similarly,

$$P_i(t) = \tilde{U}^\dagger(t) R^\dagger(\epsilon t) P_i(0) R(\epsilon t) \tilde{U}(t). \quad (20)$$

Suppose $\tilde{U}(t)$ as appeared in equation (14) and (15) is known, then from (18) and (20), $U = R\tilde{U}$ is the operator that solves the Heisenberg equations for x_i and P_i . Therefore the task of finding $U(t)$ is now reduced to the task of finding $\tilde{U}(t)$. Equations (17)-(20) embody the central point of our method, namely the tensor relations for \mathbf{x} and \mathbf{P} allow us to isolate the operator R in the evolution operator and to make full use of the vectors \mathbf{e}_1 and \mathbf{e}_2 in solving

the Heisenberg equations so that the other factor in the evolution operator can be easily found. In the following, $\tilde{U}(t)$ is determined, up to $e^{i\alpha(t)}$, by analyzing the behavior of the solutions for $\tilde{x}_i(t)$ and $\tilde{P}_i(t)$.

The equations of motion (3) and (4) can be expressed in terms of the components \tilde{x}_i and \tilde{P}_i . The approach for finding these components described below is the same as that used in Ref [7] for solving a related model. But the explanation here is self-contained. From equation (3), \mathbf{P} is determined by knowing \mathbf{x} . Substituting \mathbf{P} from equation (3) into (4) and taking the dot product of the resulting equation with \mathbf{e}_i , we obtain with the aid of (13) the following equations:

$$\ddot{\tilde{x}}_1 - \omega_c \dot{\tilde{x}}_2 = -\frac{\omega_c}{2} \sigma_2 \tilde{x}_3 + 2\sigma_1 \dot{\tilde{x}}_3 + \sigma_1^2 \tilde{x}_1 + \sigma_1 \sigma_2 \tilde{x}_2 + \dot{\sigma}_1 \tilde{x}_3, \quad (21)$$

$$\ddot{\tilde{x}}_2 + \omega_c \dot{\tilde{x}}_1 = \frac{\omega_c}{2} \sigma_1 \tilde{x}_3 + 2\sigma_2 \dot{\tilde{x}}_3 + \sigma_2^2 \tilde{x}_2 + \sigma_1 \sigma_2 \tilde{x}_1 + \dot{\sigma}_2 \tilde{x}_3, \quad (22)$$

$$\ddot{\tilde{x}}_3 + \omega^2(\tilde{x}_3 - a) = \dot{\epsilon}_3^2 \tilde{x}_3 - \frac{\omega_c}{2} \sigma_2 \tilde{x}_1 + \frac{\omega_c}{2} \sigma_1 \tilde{x}_2 - 2\sigma_1 \dot{\tilde{x}}_1 - 2\sigma_2 \dot{\tilde{x}}_2 - \dot{\sigma}_1 \tilde{x}_1 - \dot{\sigma}_2 \tilde{x}_2, \quad (23)$$

where

$$\sigma_\mu(\epsilon t) = \dot{\mathbf{e}}_\mu(\epsilon t) \cdot \mathbf{n}(\epsilon t); \quad \mu = 1, 2. \quad (24)$$

It is a useful fact that $\sigma_\mu(\epsilon t)$ is of the order of ϵ and $\dot{\sigma}_\mu$ is of the order of ϵ^2 .

The many terms in (21), (22) and (23), though may seem complicated, all have simple physical origins. Take the term $-\frac{\omega_c}{2} \sigma_2 \tilde{x}_3$ on the right hand side of (21) for example. From equation (7), we know that the frame \mathbf{e}_i is rotating with the angular velocity $\mathbf{n} \times \dot{\mathbf{n}}$. Seen from the stationary frame, \tilde{x}_3 actually moves with the velocity $\tilde{x}_3 \dot{\mathbf{n}}$, which causes a Lorentz force whose (acceleration) component along \mathbf{e}_1 is $-\omega_c \sigma_2 \tilde{x}_3$. On the other hand, the rotation of the magnetic field induces an electric field $-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$ whose acceleration along \mathbf{e}_1 is $\frac{\omega_c}{2} \sigma_2 \tilde{x}_3$. The sum of these two gives rise to the term in (21). Other terms which do not depend on B on the right hand sides are due to inertial forces associated with the rotation of the frame.

The method for treating the small terms on the right hand sides of (21)-(23) is to regard them as non-homogeneous terms and to extract their contribution to the solution iteratively, as described below. Such an iteration procedure is valid if it converges. Let us first consider the following simplified system:

$$\ddot{\tilde{x}}_1 - \omega_c \dot{\tilde{x}}_2 = -\frac{\omega_c}{2} \sigma_2 \tilde{x}_3, \quad (25)$$

$$\ddot{\tilde{x}}_2 + \omega_c \dot{\tilde{x}}_1 = \frac{\omega_c}{2} \sigma_1 \tilde{x}_3, \quad (26)$$

$$\ddot{\tilde{x}}_3 + \omega^2(\tilde{x}_3 - a) = 0. \quad (27)$$

It amounts to neglecting most of the small terms on the right hand sides of (21)-(23). The solution to this system is readily known because the terms $-\frac{\omega_c}{2} \sigma_2 \tilde{x}_3$ and $\frac{\omega_c}{2} \sigma_1 \tilde{x}_3$ are non-homogeneous terms in (25) and (26) while \tilde{x}_3 is known from

(27). If the solution to the corresponding homogeneous system of (25) and (26) and the solution to (27) are denoted as $\tilde{x}_i^{(0)}(t)$, then

$$\tilde{x}_\mu(t) = \tilde{x}_\mu^{(0)}(t) + \frac{1}{2} \int_0^t \sigma_\mu(\epsilon t') \tilde{x}_3^{(0)}(t') dt', (\mu = 1, 2) \quad (28)$$

while $\tilde{x}_3(t) = \tilde{x}_3^{(0)}(t)$.

The general iteration procedure for treating (21)-(23) is similar. By substituting $\tilde{x}_i^{(0)}(t)$ into the right hand sides of (21)-(23), we will obtain the first order approximation to the solution and so forth. However, the terms appeared in (21)-(23) but dropped in (25)-(27) do not contribute to the solution in the adiabatic limit of $|\frac{\epsilon}{\omega - \omega_c}| \rightarrow 0$, $\frac{\epsilon}{\omega} \rightarrow 0$, and $\frac{\epsilon}{\omega_c} \rightarrow 0$. This is just the usefulness of the parallel transported vectors \mathbf{e}_μ , namely, the use of condition (13) has resulted in a simplification. So the system (21)-(23) and the system (25)-(27) are equivalent in the adiabatic limit. The adiabatic limit defined above is reasonable and the case of $\omega = \omega_c$ is excluded. Otherwise, there will be further degeneracy which is not of particular concern to the magnetic field problem.

Therefore, in the adiabatic limit, (28) is the solution for $\tilde{x}_\mu(t)$ as appeared in (21)-(23). Now $\tilde{x}_3^{(0)}(t) = a + [x_3(0) - a] \cos \omega t + \frac{\dot{x}_3(0)}{\omega} \sin \omega t$; the two oscillating terms, when substituted into (28), vanish in the limit of $\epsilon \rightarrow 0$, therefore we have

$$\tilde{x}_\mu(t) = \tilde{x}_\mu^{(0)}(t) - \frac{a}{2} \int_{\mathbf{n}(0)}^{\mathbf{n}(\epsilon t)} \mathbf{e}_\mu \cdot d\mathbf{n}, \quad (29)$$

where use has been made of equation (24). In the above, the result of the integral is path-dependent, because \mathbf{e}_μ is dependent on the path of \mathbf{n} . It is noticeable that although the shifting of the orbit as represented by the integral is due to the electromagnetic force (as in equation (25) and (26)), the result is independent on the magnitude of the magnetic field. By now, we have solved the Heisenberg equations (3) and (4) in the adiabatic limit by making use of the two parallel transported unit vectors \mathbf{e}_μ .

Now we turn to the problem of finding the operator \tilde{U} in equation (14) and (15) that can realize (29). Observe that although the integral in (29) is finite, its derivative with respect to time is of the order of ϵ which goes to zero when $\epsilon \rightarrow 0$. Therefore, the velocity operator satisfies: $\dot{\tilde{x}}_i(t) = \dot{\tilde{x}}_i^{(0)}(t)$. It follows from this that the Heisenberg picture Hamiltonian (different from the Schrödinger picture Hamiltonian because of its time dependence) of the system (2) is an invariant in the adiabatic limit. Therefore equation (29) should be seen as embodying the adiabatic theorem in a concrete way that is known only through solving the equations. Now it is important to observe that the generator $\mathbf{P}(0)$ of the ordinary translation does not commute with the velocity operator. However, it is easy to show that $[P_i(0) + \frac{q}{c} A_i(0), P_j(0) - \frac{q}{c} A_j(0)] = 0$. Namely, a translation generated by $\mathbf{P}(0) + \frac{q}{c} \mathbf{A}(0)$ can leave the velocity operator invariant. Such a

translation is known as a magnetic translation and is studied in Ref [4] and [5]. The product of the dynamical operator $e^{-\frac{i}{\hbar}H(0)t}$ which evolves $\tilde{x}_i(0)$ and $\tilde{P}_i(0)$ to $\tilde{x}_i^{(0)}(t)$ and $\tilde{P}_i^{(0)}(t)$ respectively, and a magnetic translation operator which preserves $\dot{\tilde{x}}_i^{(0)}(t)$ but shifts $\tilde{x}_i^{(0)}(t)$ to $\tilde{x}_i(t)$ qualifies as \tilde{U} :

$$\tilde{U}(t) = e^{-\frac{i}{\hbar}H(0)t}M(\epsilon t), \quad (30)$$

$$M(\epsilon t) = \exp \left(-\frac{i}{\hbar} \left(-\frac{a}{2} \int_{\mathbf{n}(0)}^{\mathbf{n}(\epsilon t)} \mathbf{e}_\mu \cdot d\mathbf{n} \right) \mathbf{e}_\mu(0) \cdot [\mathbf{P}(0) + \frac{q}{c}\mathbf{A}(0)] \right). \quad (31)$$

With this, we determined the evolution operator up to a phase factor in the following form:

$$U(t) = R(\epsilon t)M(\epsilon t)D(t) \quad (32)$$

where we used the fact that $M(\epsilon t)$ and $D(t) = e^{-\frac{i}{\hbar}H(0)t}$ commute, and the operator R is given by equation (19).

The expression for M as given by (31) can be verified directly by checking equations (14) and (15). Intuitively, the magnetic translation should be thought of as happening in the parallel transported frame. Then through the operator R , we know how the quantum system evolves in the frame specified by the Hamiltonian (what we call the stationary frame). Notice that the displacement vector in equation (31) is

$$\mathbf{d}(\epsilon t) = \left(-\frac{a}{2} \int_{\mathbf{n}(0)}^{\mathbf{n}(\epsilon t)} \mathbf{e}_\mu(\epsilon t') \cdot d\mathbf{n} \right) \mathbf{e}_\mu(0), \quad (33)$$

which agrees with the fact that inside the parallel transported frame, the coordinate axes should be seen as fixed, i.e., $\mathbf{e}_i(0)$. By now, U has been written as the product of a path-dependent geometrical operator and a dynamical operator.

As mentioned in the introduction, the operator U is not unique. Obviously, the exponential in (31) may be replaced by the corresponding path-ordered exponential. It gives a different U that gives the same $x_i(t)$ and $P_i(t)$. The magnetic translation and its path-ordered exponential alternative are different because magnetic translations along different directions do not commute. In the next section it is shown that it is the path-ordered magnetic translation that enters into the evolution operator. Our purpose below is to find the relation between magnetic translation and the corresponding path-ordered magnetic translation by using some simple properties of the magnetic translation operator[4].

Denote the path-ordered exponential as M_P . We have

$$M_P(\epsilon t) = e^{i\phi_P(\epsilon t)}M(\epsilon t). \quad (34)$$

It has the advantage of easy differentiation, i.e.,

$$\frac{d}{dt}M_P(\epsilon t) = \left(\frac{ia}{2\hbar} (\mathbf{e}_\mu \cdot \dot{\mathbf{n}}) \mathbf{e}_\mu(0) \cdot [\mathbf{P}(0) + \frac{q}{c}\mathbf{A}(0)] \right) M_P(\epsilon t). \quad (35)$$

To determine $\phi_P(\epsilon t)$ in equation (34), consider a magnetic translation corresponding to the displacement \mathbf{d} ,

$$M(\mathbf{d}) = \exp \left(-\frac{i}{\hbar} \mathbf{d} \cdot [\mathbf{P}(0) + \frac{q}{c} \mathbf{A}(0)] \right). \quad (36)$$

It is straight forward to verify that

$$M(\mathbf{d}_2)M(\mathbf{d}_1) = M(\mathbf{d}_1 + \mathbf{d}_2) \exp \left(-\frac{i}{2} (\mathbf{d}_1 \times \mathbf{d}_2) \cdot \frac{qB\mathbf{n}(0)}{\hbar c} \right). \quad (37)$$

This relation is the same as equation (9) in Ref [4]. (In [4], $-e$ is the charge for the electron, i.e., $q = -e$. Also, we use M instead of T for the magnetic translation operator since the latter is reserved for the time of the adiabatic evolution.) Observe that $\frac{1}{2}(\mathbf{d}_1 \times \mathbf{d}_2) \cdot B\mathbf{n}(0)$ is the flux through the triangle formed by \mathbf{d}_1 and \mathbf{d}_2 with the tail of \mathbf{d}_2 sitting on tip of \mathbf{d}_1 . So it follows from equation (37) that for a sequence of magnetic translations corresponding to displacement around a closed path C , $M_P = \exp \left(-\frac{iq}{\hbar c} \Phi_C \right)$, where Φ_C is the magnetic flux through the loop C . (In [4], the phase was erroneously written as “flux($\frac{e}{2\hbar c}$)”, it should obviously be “flux($\frac{e}{\hbar c}$)”.) For an open path, it follows from (37), the flux considered is through the loop formed by the curve $\mathbf{d}(\epsilon t')$ with t' going from 0 to t and the straight line pointing from $\mathbf{d}(\epsilon t)$ to $\mathbf{0}$. The area enclosed by this loop is known because \mathbf{d} is given by equation (33). The result for $\phi_P(\epsilon t)$ is:

$$\phi_P(\epsilon t) = -\left(\frac{q}{\hbar c}\right) \frac{Ba^2}{4} \left(\int_0^t \sigma_2(\epsilon t') dt' \int_0^{t'} \sigma_1(\epsilon t'') dt'' - \frac{1}{2} \int_0^t \sigma_1(\epsilon t') dt' \int_0^t \sigma_2(\epsilon t') dt' \right). \quad (38)$$

By the expression for σ_μ in equation (24), $\phi_P(\epsilon t)$ is path-dependent just as expected.

3 The Quantum Evolution Operator in a Factorised Form

The operator U obtained in the last section contains all the information from the Heisenberg equations. From the fundamentals of quantum theory, we know that the Heisenberg picture and the Schrödinger picture are equivalent. If we consider a whole quantum system as governed by an adiabatically changing Hamiltonian, then an Abelian phase factor is of course not physically observable [8]. However if we think of the adiabatic Hamiltonian as governing one component of a quantum system that is later to interfere with the other component that has gone through a different evolution, then the result is dependent on the phase content of each of the components. An Abelian phase factor therefore is observable in an interference experiment. But it is important to bear in mind in this case that the evolution of the other component is not governed by the

adiabatic Hamiltonian. So far as a complete quantum system is concerned, the Heisenberg picture and the Schrödinger picture should contain equal amounts of physical information. Because of the ambiguity in the choice of U as mentioned at the end of the introduction, it is impossible to determine completely the quantum evolution operator (and therefore the Berry phase factor) through the Heisenberg picture.

The purpose of this section is to eliminate the ambiguity by making use of the Schrödinger picture so that the quantum evolution operator is determined completely. Let the evolution operator be \mathcal{U} . Then \mathcal{U} is related to U by an Abelian phase factor,

$$\mathcal{U} = e^{i\alpha}U = e^{i\alpha}RMD = e^{i\alpha}e^{-i\phi_P}RM_PD, \quad (39)$$

where α is a real number. In the following, it is shown that $\alpha = \phi_P(\epsilon t)$.

Consider an eigenstate $|\Psi(0)\rangle$ of the initial Hamiltonian $H(0)$ that is a direct product of an eigenstate of the harmonic oscillator along $\mathbf{n}(0)$ and a wave function describing motion perpendicular to $\mathbf{n}(0)$:

$$|\Psi(0)\rangle = |\Psi_\perp(0)\rangle |Osc(0)\rangle. \quad (40)$$

Such an eigenstate has the following property:

$$\langle \Psi(0) | \tilde{x}_3(t) | \Psi(0) \rangle = a, \quad (41)$$

$$\langle \Psi(0) | \dot{\tilde{x}}_i(t) | \Psi(0) \rangle = 0. \quad (42)$$

All the $|\Psi(0)\rangle$'s form a complete set of eigenfunctions. At a later time, $|\Psi(0)\rangle$ evolves into:

$$|\Psi(t)\rangle = \mathcal{U}(\epsilon t) |\Psi(0)\rangle = e^{i\alpha(\epsilon t)} R(\epsilon t) M(\epsilon t) e^{-\frac{i}{\hbar} H(0)t} |\Psi(0)\rangle. \quad (43)$$

Since we already proved the adiabatic theorem in the previous section for the system (2), when substituting $|\Psi(t)\rangle$ into the Schrödinger equation,

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = H(\mathbf{n}(\epsilon t)) |\Psi(t)\rangle, \quad (44)$$

we get the following condition:

$$\dot{\alpha} = i\langle \Psi(0) | U^\dagger (\dot{R}R^\dagger) U | \Psi(0) \rangle + i\langle \Psi(0) | \tilde{U}^\dagger (\dot{M}M^\dagger) \tilde{U} | \Psi(0) \rangle. \quad (45)$$

The purpose is to determine α . Observed that $\dot{\alpha}$ is necessarily of the order of ϵ . But upon integration ($t \in [0, 1/\epsilon]$) it has a finite result and indeed α is path-dependent. By equation (19),

$$\begin{aligned} iU^\dagger \dot{R}R^\dagger U &= iU^\dagger (\mathbf{n} \times \dot{\mathbf{n}}) \cdot \mathbf{e}_m(0) \frac{-iL_m(0)}{\hbar} U, \\ &= U^\dagger \frac{1}{\hbar} (\mathbf{n} \times \dot{\mathbf{n}}) \cdot \mathbf{e}_m(\epsilon t) (\mathbf{x}(0) \times \mathbf{P}(0)) \cdot \mathbf{e}_m(\epsilon t) U, \\ &= \frac{1}{\hbar} (\mathbf{n} \times \dot{\mathbf{n}}) \cdot \mathbf{e}_m(\epsilon t) (\mathbf{x}(t) \times \mathbf{P}(t)) \cdot \mathbf{e}_m(\epsilon t). \end{aligned} \quad (46)$$

While according to equations (5) and (6), $\mathbf{x}(t) \times \mathbf{P}(t) = \tilde{x}_i(t)\tilde{P}_j(t)\mathbf{e}_i(\epsilon t) \times \mathbf{e}_j(\epsilon t)$. Since $\tilde{x}_i(t)$ and $\tilde{P}_i(t)$ are found in the previous section in terms of $\tilde{x}_i^{(0)}(t)$ and the displacement vector, the calculation of $i\langle U^\dagger(\dot{R}R^\dagger)U \rangle$ is reduced to the calculation of terms such as $\langle \tilde{x}_3(t)\tilde{P}_1(t) \rangle$, $\langle \tilde{P}_3(t)\tilde{x}_1(t) \rangle$, etc., with coefficients expressible in terms of $\sigma_\mu(\epsilon t)$ on account of equation (24). Observe from (45) and (46) that $\dot{\alpha}$ is already of ϵ order due to $\dot{\mathbf{n}}$, so we do not need to keep ϵ order terms from the products $\tilde{x}_3(t)\tilde{P}_1(t)$, $\tilde{P}_3(t)\tilde{x}_1(t)$, etc., because they make no contribution to α in the adiabatic limit. The result is:

$$i\langle \Psi(0) | U^\dagger(\dot{R}R^\dagger)U | \Psi(0) \rangle = \frac{qaB}{2c\hbar} \left(\sigma_1 \langle d_2 + \tilde{x}_2^{(0)}(t) \rangle - \sigma_2 \langle d_1 + \tilde{x}_2^{(0)}(t) \rangle \right). \quad (47)$$

By making use of (34) and (35), the second term on the right hand side of (45) can also be calculated in a similar way with the following result:

$$i\langle \Psi(0) | \tilde{U}^\dagger(\dot{M}M^\dagger)\tilde{U} | \Psi(0) \rangle = \dot{\phi}_P - \frac{qaB}{2c\hbar} \left(\sigma_1 \langle d_2 + \tilde{x}_2^{(0)}(t) \rangle - \sigma_2 \langle d_1 + \tilde{x}_2^{(0)}(t) \rangle \right). \quad (48)$$

Therefore, with (45), (47), (48) and the initial condition $\alpha(0) = 0$, we have:

$$\alpha(\epsilon t) = \phi_P(\epsilon t). \quad (49)$$

Since $\alpha(\epsilon t)$ does not depend on the choice of $|\Psi(0)\rangle$, it is the same for all initial states. With this result and equation (39), the quantum evolution operator is determined to be:

$$\mathcal{U}(\epsilon t) = R(\epsilon t)M_P(\epsilon t)D(t), \quad (50)$$

where $M_P(\epsilon t)$ is given by equations (34) and (38) in terms of the magnetic translation operator. Because magnetic translations along two different directions do not commute, the Berry phase factor, if $\mathbf{n}(1) = \mathbf{n}(0)$, is essentially non-Abelian.

4 Remarks and Possible Experiment

In the following, some remarks are given. Experimental verification of the result (50) is also proposed.

(I) The infinite degeneracy requires solving the Heisenberg equations explicitly in order to take into account the small perturbations in the rotating frame. Otherwise, say a confining potential is present in the two dimensional plane and therefore eliminates the infinite degeneracy, the small perturbations would be averaged out. These are quite different perturbations.

(II) Equation (29) also sheds light on the situation of no confinement in the direction of the magnetic field. For such a case, motion along the direction of the magnetic field is not bounded. In the adiabatic limit (if such a meaningful limit exists), $\tilde{x}_3 \sim a$ would change greatly with time (because $t \in [0, T]$), leading to complicated behavior in the $\tilde{x}_1\tilde{x}_2$ plane which may or may not obey any adiabatic theorem. At least, none of the terms in the set of equations (21)-(23) can be neglected.

(III) The operator R as given by formula (19) is determined (up to any time-dependent numerical phase factor) by the tensor relations $E_{ij}(\epsilon t)x_j(0) = R^\dagger(\epsilon t)x_i(0)R(\epsilon t)$ and $E_{ij}(\epsilon t)P_j(0) = R^\dagger(\epsilon t)P_i(0)R(\epsilon t)$. Notice especially that $-i$ in (19) cannot be replaced by i ; for a cyclic change ($\mathbf{n}(1) = \mathbf{n}(0)$) the latter results in “ $\exp(\frac{i}{\hbar}\Omega\mathbf{L}(0) \cdot \mathbf{n}(0))$ ” while the correct result that follows from (19) is $R(1)=\exp(-\frac{i}{\hbar}\Omega\mathbf{L}(0) \cdot \mathbf{n}(0))$, which is equivalent to the solid angle result well-known from the spin case [2] except that due to the existence of magnetic translation, $\mathbf{L}(t) \cdot \mathbf{n}(\epsilon t)$ is not an adiabatic invariant now. (Ω is the solid angle equal to the oriented area enclosed by the loop of \mathbf{n} on the two-sphere.)

(IV) The magnetic translation is a natural generalization [4], [5] of ordinary translation when a magnetic field is present. The result (50) says that by rotating the magnetic field and the confining potential, a wave packet will be displaced in the two dimensional plane perpendicular to the magnetic field direction in addition to a rotation that follows that direction. The amount of displacement is predicted by (33). The transverse position of the particle at the end of a cycle may be detected by withdrawing the confining potential and applying an accelerating electric field along the magnetic field direction so that the particle can move out of the solenoid to reach a detector.

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